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Marginal and correlation distribution functions in the squeezed-states representation

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Abstract. Here we consider the Husimi function (HF) P for the squeezed states and calculate the marginal and correlation distribution functions (MDFs and CDFs) when P is projected onto the photon-number states. According to the value of the squeezing parameter one verifies the occurence of oscillations and beats as already appointed in the literature. We verify that these phenomena are entirely contained in the correlation function. In particular, we show that since the Husimi and its MDFs satisfy partial differential equations where the squeeze parameter plays the role of time, the solutions (the squeezed functions obtained from 'initial' unsqueezed functions) can be expressed by means of kernels responsible for the 'propagation' of squeezing. From the calculational point of view, this method presents advantages for calculating the MDFs (compared with a direct integration over one of the two phase-space variables of P) since one can use the symmetry properties of the differential equations.

1. Introduction

At the end of the 1980s, the oscillations of the photon distribution function of high-energy squeezed and correlated states were discovered in [1,2]; the authors of [1] studied the oscillatory behaviour of the single-mode squeezed-state Husimi fuction (HF) projected into photon-number states, $P_n(p,q;\lambda,\phi) = |\langle n|p,q;\lambda,\phi\rangle|^2$, where p and q are the space variables associated with the two quadratures of a monochromatic electromagnetic (EM) field, λ is the squeezing parameter and ϕ is a rotation angle in phase space. They suggested that for $p=0$, $q=7\sqrt{2}$, $\phi=0$ and a fixed $\lambda=21$, such oscillatory behaviour can be explained in terms of quantum interference effects and were taken as a signature of a nonclassical state. More recently, Gagen [3] generalized this study to incorporate interference structures in the Bohr–Sommerfeld trajectories associated with a superposition of quantum states. On the other hand, Dutta *et al* [4] verified an additional feature present in $P_n(p,q;\lambda,\phi)$, when plotted as function of n: this distribution exhibits the structure of beats (collapses and revivals) at large values of $n(\geqslant 10)$ for $\lambda=201$, $p^2+q^2=98$ and $\phi\approx\pi/2$. Compared with the value λ used in [1], the high value for λ is crucial for the occurence of beats. These oscillations were attributed to interference effects in phase space [4]; however, since a detailed explanation concerning the beats has not been presented until recently, we judge that they deserve a deeper investigation.

Recently, Chountasis and Vourdas [5] showed that the Weyl function is an important tool for quantum interference effects. In particular, they studied the Wigner and Weyl functions for a superposition of m quantum states $|s_i\rangle$, where each function is decomposed into diagonal and nondiagonal terms, where the nondiagonal term is responsible for the interference between the states $|s_i\rangle$. Adopting a different approach and using the formalism developed in [6], and in order to shed more light on the origin of the beats predicted in [4], here we decompose the squeezed state HF into three functions: the two marginal distribution functions (MDFs) (one for p and the other for q) and the correlation distribution function (CDF). Our results corroborate the phase-space interference concept and complement the graphical treatment proposed by Mandal [7].

This paper is organized as follows. In section 2 we discuss the solution of the differential pseudo-diffusion equation (see [6]), and in section 3 we define the phase-space MDF and CDF. The formal and numerical results are given in section 4, where we show that oscillations and beats are present in the CDF. Section 5 contains our summary and conclusions. Two appendices are also presented, containing calculational details. Appendix A contains the main steps to calculate the MDFs by direct integration, and in appendix B we calculate the CDF.

2. The pseudo-diffusion equation and its solutions

The mapping of the statistical operator $\boldsymbol{\rho}$ (describing a state of the EM field) on the squeezed-states representation (SSR) permits us to write the HF P as follows [6]:

$$P(p,q;\lambda,\phi) = \mathrm{Tr}[\boldsymbol{\rho}\boldsymbol{\Pi}(p,q;\zeta)] = \mathrm{Tr}[\boldsymbol{\rho}_r\boldsymbol{\Pi}(p_r,q_r;\lambda)] = P_r(p_r,q_r;\lambda) \qquad (1)$$

where

$$\boldsymbol{\Pi}(p,q;\zeta) = |pq;\zeta\rangle\langle pq;\zeta| = \boldsymbol{R}(\phi/2)\boldsymbol{\Pi}(p_r,q_r;y)\boldsymbol{R}^{\dagger}(\phi/2) \qquad (2)$$

is a projection operator, $\boldsymbol{R}(\phi/2) = \exp\left(\frac{\mathrm{i}\phi}{2}\boldsymbol{a}^{\dagger}\boldsymbol{a}\right)$ is the rotation operator, $q_r = q\cos(\phi/2) + p\sin(\phi/2)$ and $p_r = p\cos(\phi/2) - q\sin(\phi/2)$ are the rotated phase-space variables expressed in terms of the old ones, $\boldsymbol{\rho}_r = \boldsymbol{R}^{\dagger}(\phi/2)\boldsymbol{\rho}\boldsymbol{R}(\phi/2)$ and $\lambda \equiv \mathrm{e}^{-2y}(0 < \lambda < \infty)$. Now, if one considers the mixed state $\boldsymbol{\rho} = \sum_{n=0}^{\infty} p_n|n\rangle\langle n|$, diagonal in the Fock basis, $\boldsymbol{\rho}_r$ will be invariant under rotations since $\boldsymbol{R}(\phi/2)|n\rangle = \mathrm{e}^{\mathrm{i}n\phi/2}|n\rangle$. Consequently, the associated HF is given by $P(p,q;\lambda,\phi) = P(p_r,q_r;\lambda)$. This relation is useful in the sense that it is sufficient to consider the calculation of the unrotated distribution $P(p,q;\lambda)$, with variables changed from (p,q) to (p_r,q_r) in the final result, respectively. The number state $|n\rangle\langle n|$ is a typical example where this relation can be used. In [6] we demonstrated that $P(p,q;\lambda,\phi)$ satisfies the partial differential equation

$$\boldsymbol{\Gamma}(p,q;\lambda,\phi)P(p,q;\lambda,\phi) = 0 \qquad (3)$$

where

$$\boldsymbol{\Gamma}(p,q;\lambda,\phi) = \frac{\partial}{\partial\lambda} - \frac{1}{4\lambda^2}\left\{[\lambda^2\cos^2(\phi/2) - \sin^2(\phi/2)]\frac{\partial^2}{\partial p^2}\right.$$
$$\left. + [\lambda^2\sin^2(\phi/2) - \cos^2(\phi/2)]\frac{\partial^2}{\partial q^2} - (\lambda^2+1)\sin\phi\frac{\partial^2}{\partial q\partial p}\right\} \qquad (4)$$

is a linear differential operator. For $\phi = 0$, equation (3) is similar to the diffusion equation in two dimensions where the parameter λ plays the role of time. In this situation, since the diffusion coefficients have opposite signs, the equation describes a diffusive (infusive) process in the $p(q)$ variable. For this reason, it has been called the *pseudo-diffusion equation* [8, 9].

Here we consider the formal solution of equation (3), obtained by the Fourier transform (FT) method, written as an integral equation with the kernel depending on the squeeze and rotation parameters,

$$\begin{aligned} P(p,q;\lambda,\phi) &= \int_{-\infty}^{\infty} \frac{\mathrm{d}\xi\,\mathrm{d}\eta}{2\pi} \mathrm{e}^{\mathrm{i}(\eta p-\xi q)} K(\xi,\eta;\lambda,\phi)\widetilde{P}(\xi,\eta) \\ &= \int_{-\infty}^{\infty} \frac{\mathrm{d}\xi\,\mathrm{d}\eta}{2\pi} \mathrm{e}^{\mathrm{i}(\eta p_r-\xi q_r)} K(\xi,\eta;\lambda,0)\widetilde{P}_r(\xi,\eta) \\ &= P_r(p_r,q_r;\lambda). \end{aligned} \tag{5}$$

The kernel is

$$\begin{aligned} K(\xi,\eta;\lambda,\phi) = \exp\biggl(&-\frac{\lambda-1}{4\lambda}\{[\lambda\sin^2(\phi/2)-\cos^2(\phi/2)]\xi^2 \\ &+[\lambda\cos^2(\phi/2)-\sin^2(\phi/2)]\eta^2+(\lambda+1)\sin\phi\,\xi\eta\}\biggr) \end{aligned} \tag{6}$$

and $\widetilde{P}(\xi,\eta)$ is the FT of the HF $P(p,q)$ for $\lambda=1$ and $\phi=0$. We notice that $K(\xi,\eta;\lambda,\phi)$ is an *unbounded* function, responsible for the squeezing 'propagation' of an 'initial' function $P(p,q)$ to $P(p,q;\lambda,\phi)$ for any values of λ and ϕ in their domain. Thus the existence of a 'propagated' $P(p,q;\lambda,\phi)$ depends on the functional form of $\widetilde{P}(\xi,\eta)$, since the integral in the first line in (5) for $\widetilde{P}(\xi,\eta)=$ constant does not exist. From the pseudo-diffusion equation (3) and the linear differential operator $\boldsymbol{\Gamma}$, equation (5) shows the following symmetry properties:

$$P(p,q;\lambda,\phi) = P(q,-p;\lambda,\phi\pm\pi) = P(q,p;\lambda^{-1},-\phi) \tag{7}$$

and

$$K(\xi,\eta;\lambda,\phi) = K(\eta,-\xi;\lambda,\phi\pm\pi) = K(\eta,\xi;\lambda^{-1},-\phi). \tag{8}$$

Now, our aim is to show that the Glauber–Sudarshan distribution $P^c(p,q;\lambda,\phi)$ and HF $P(p,q;\lambda,\phi)$ are related by

$$P^c(p,q;\lambda,\phi) = \boldsymbol{\Lambda}(p,q;\lambda,\phi)P(p,q;\lambda,\phi) = P(p,q;-\lambda,\phi) \tag{9}$$

with

$$\boldsymbol{\Lambda}(p,q;\lambda,\phi) \equiv \exp\left[-\frac{1}{2}\left(\lambda\frac{\partial^2}{\partial p_r^2}+\lambda^{-1}\frac{\partial^2}{\partial q_r^2}\right)\right] \tag{10}$$

which can also be written as

$$\begin{aligned} \boldsymbol{\Lambda}(p,q;\lambda,\phi) = \exp\biggl\{-\frac{1}{2\lambda}\biggl[&[\lambda^2\cos^2(\phi/2)+\sin^2(\phi/2)]\frac{\partial^2}{\partial p^2} \\ &+[\lambda^2\sin^2(\phi/2)+\cos^2(\phi/2)]\frac{\partial^2}{\partial q^2}-(\lambda^2-1)\sin\phi\frac{\partial^2}{\partial p\partial q}\biggr]\biggr\} \end{aligned}$$

since

$$\begin{aligned} \frac{\partial^2}{\partial p_r^2} &= \sin^2(\phi/2)\frac{\partial^2}{\partial q^2}+\cos^2(\phi/2)\frac{\partial^2}{\partial p^2}-\sin\phi\frac{\partial^2}{\partial q\partial p} \\ \frac{\partial^2}{\partial q_r^2} &= \cos^2(\phi/2)\frac{\partial^2}{\partial q^2}+\sin^2(\phi/2)\frac{\partial^2}{\partial p^2}+\sin\phi\frac{\partial^2}{\partial q\partial p}. \end{aligned}$$

Applying the differential operator $\boldsymbol{\Lambda}$ on (5), we get

$$\boldsymbol{\Lambda}(p,q;\lambda,\phi)P(p,q;\lambda,\phi) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\xi\,\mathrm{d}\eta}{2\pi}[\boldsymbol{\Lambda}(p,q;\lambda,\phi)\mathrm{e}^{\mathrm{i}(\eta p_r-\xi q_r)}]K(\xi,\eta;\lambda,0)\widetilde{P}_r(\xi,\eta)$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}\xi\,\mathrm{d}\eta}{2\pi} \mathrm{e}^{\mathrm{i}(\eta p_r - \xi q_r)} \underbrace{\mathrm{e}^{\frac{1}{2}(\lambda^{-1}\xi^2 + \lambda\eta^2)} K(\xi,\eta;\lambda,0)}_{K(\xi,\eta;-\lambda,0)} \widetilde{P}_r(\xi,\eta) \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}\xi\,\mathrm{d}\eta}{2\pi} \mathrm{e}^{\mathrm{i}(\eta p_r - \eta q_r)} K(\xi,\eta;-\lambda,0) \widetilde{P}_r(\xi,\eta) \\
&= P(p,q;-\lambda,\phi). \qquad (11)
\end{aligned}
$$

The second equality is obtained using the following relation:

$$
\begin{aligned}
\mathbf{\Lambda}(p,q;\lambda,\phi)\mathrm{e}^{\mathrm{i}(\eta p_r - \xi q_r)} &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} \left(\lambda^{-1} \frac{\partial^2}{\partial q_r^2} + \lambda \frac{\partial^2}{\partial p_r^2} \right)^k \mathrm{e}^{\mathrm{i}(\eta p_r - \xi q_r)} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{1}{2}(\lambda^{-1}\xi^2 + \lambda\eta^2) \right]^k \mathrm{e}^{\mathrm{i}(\eta p_r - \xi q_r)} \\
&= \mathrm{e}^{\frac{1}{2}(\lambda^{-1}\xi^2 + \lambda\eta^2)} \mathrm{e}^{\mathrm{i}(\eta p_r - \xi q_r)}
\end{aligned}
$$

and, from the definition of the kernel $K(\xi,\eta;\lambda,0)$, we conclude that

$$
\mathrm{e}^{\frac{1}{2}(\lambda^{-1}\xi^2 + \lambda\eta^2)} K(\xi,\eta;\lambda,0) = \mathrm{e}^{\frac{1}{4}(\lambda+1)(\eta^2 + \lambda^{-1}\xi^2)} = K(\xi,\eta;-\lambda,0).
$$

Thus, the distribution function $P^c(p,q;\lambda,\phi)$ is obtained directly by changing the signal of the squeezing parameter $\lambda \to -\lambda$ in the HF. Consequently, this result shows that $P_n^c(p,q;\lambda,\phi)$ does not exist as a bounded function for the number states, however, it exists as an ultradistribution. Equation (9) is a generalization of previous results obtained in [9, 10] for $\phi = 0$.

3. MDFs and CDFs

The HF $P(p,q;\lambda,\phi)$ can be written as a sum of two terms: the first is the product of the two MDFs, in q and p, and describes the noncorrelated part; the second term is the CDF and contains the phase-space correlations [6]. So, the HF (1) can be written as

$$
P(p,q;\lambda,\phi) = R(p;\lambda,\phi)Q(q;\lambda,\phi) + C(p,q;\lambda,\phi) \qquad (12)
$$

where

$$
Q(q;\lambda,\phi) = \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{\sqrt{2\pi}} P(p,q;\lambda,\phi) \qquad (13)
$$

and

$$
R(p;\lambda,\phi) = \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\sqrt{2\pi}} P(p,q;\lambda,\phi) \qquad (14)
$$

are the MDFs, and $C(p,q;\lambda,\phi)$ is the CDF. However, the calculation of (13) and (14) by direct integration displays difficulties when $\phi \neq 0$ (see appendix A). Thus, the aim of this section is to show that expressions for the MDFs can be obtained in a much simpler way by using the formalism of section 2.

Substituting the right-hand side (RHS) of the first line of equation (5) into the integrands of equations (13) and (14), and then carrying out the integrations we get

$$
Q(q;\lambda,\phi) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\xi}{\sqrt{2\pi}} \mathrm{e}^{-\mathrm{i}\xi q} k_Q(\xi;\lambda,\phi) \widetilde{Q}(\xi) \qquad (15)
$$

and

$$
R(p;\lambda,\phi) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\eta}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}\eta p} k_R(\eta;\lambda,\phi) \widetilde{R}(\eta) \qquad (16)
$$

where

$$k_Q(\xi;\lambda,\phi)=\exp\left\{-\frac{\lambda-1}{4\lambda}[\lambda\sin^2(\phi/2)-\cos^2(\phi/2)]\xi^2\right\} \tag{17}$$

and

$$k_R(\eta;\lambda,\phi)=\exp\left\{-\frac{\lambda-1}{4\lambda}[\lambda\cos^2(\phi/2)-\sin^2(\phi/2)]\eta^2\right\} \tag{18}$$

are the reduced kernels responsible for the 'propagation' of the squeezing. The functions $\widetilde{Q}(\xi)$ and $\widetilde{R}(\eta)$ are the respective FTs of the HFs $Q(q)$ and $R(p)$ for $\lambda=1$ (absence of squeezing). Futhermore, equations (15) and (16) are solutions of the partial differential equations

$$\left[\frac{\partial}{\partial\lambda}-\frac{\lambda^2\sin^2(\phi/2)-\cos^2(\phi/2)}{4\lambda^2}\frac{\partial^2}{\partial q^2}\right]Q(q;\lambda,\phi)=0 \tag{19}$$

$$\left[\frac{\partial}{\partial\lambda}-\frac{\lambda^2\cos^2(\phi/2)-\sin^2(\phi/2)}{4\lambda^2}\frac{\partial^2}{\partial p^2}\right]R(p;\lambda,\phi)=0. \tag{20}$$

In analogy to (8), the reduced kernels k_Q and k_R have the symmetry properties

$$k_{Q(R)}(x;\lambda,\phi)=k_{R(Q)}(x;\lambda^{-1},\phi)=k_{R(Q)}(x;\lambda,\pi\pm\phi) \tag{21}$$

which reflect directly into the MDFs,

$$\begin{aligned}Q(q;\lambda,\phi)&=R(q;\lambda^{-1},\phi)=R(q;\lambda,\pi\pm\phi)\\ R(p;\lambda,\phi)&=Q(p;\lambda^{-1},\phi)=Q(p;\lambda,\pi\pm\phi).\end{aligned} \tag{22}$$

Consequently, the calculation of $Q(q;\lambda,\phi)$ is sufficient for determining the function $R(p;\lambda,\phi)$, and vice versa.

Now we analyse the structure of the kernel (6), which can be factorized as

$$K(\xi,\eta;\lambda,\phi)=k_Q(\xi;\lambda,\phi)k_R(\eta;\lambda,\phi)k_C(\xi,\eta;\lambda,\phi) \tag{23}$$

where the first two factors on the RHS (reduced kernels) 'propagate' the initial HF in an independent way, i.e., if in the 'initial' ($\lambda=1$) HF the phase-space variables are not correlated, they will remain as such for any other value of λ. The factor

$$k_C(\xi,\eta;\lambda,\phi)=\exp\left[-\left(\frac{\lambda^2-1}{4\lambda}\sin\phi\right)\xi\eta\right] \tag{24}$$

introduces additional (or new) correlations into an 'initial' HF when $\phi\neq n\pi$, with $n\in\mathbb{N}$. Otherwise, we obtain $k_C=1$ and $K(\xi,\eta;\lambda,n\pi)=k_Q(\xi;\lambda,n\pi)k_R(\eta;\lambda,n\pi)$, one reduced kernel for each variable.

As a consequence of factorization (23) it is interesting to rewrite the CDF $C(p,q;\lambda,\phi)$ as a sum of two terms,

$$C(p,q;\lambda,\phi)=C^{(1)}(p,q;\lambda,\phi)+C^{(2)}(p,q;\lambda,\phi) \tag{25}$$

defined as

$$C^{(1)}(p,q;\lambda,\phi)=\int_{-\infty}^{\infty}\frac{\mathrm{d}\xi\,\mathrm{d}\eta}{2\pi}\mathrm{e}^{\mathrm{i}(\eta p-\xi q)}K(\xi,\eta;\lambda,\phi)\widetilde{C}(\xi,\eta) \tag{26}$$

and

$$\begin{aligned}C^{(2)}(p,q;\lambda,\phi)=\int_{-\infty}^{\infty}&\frac{\mathrm{d}\xi\,\mathrm{d}\eta}{2\pi}\mathrm{e}^{\mathrm{i}(\eta p-\xi q)}k_Q(\xi;\lambda,\phi)k_R(\eta;\lambda,\phi)\\ &\times[k_C(\xi,\eta;\lambda,\phi)-1]\widetilde{R}(\eta)\widetilde{Q}(\xi).\end{aligned} \tag{27}$$

Here we assume that the HF contains 'initial' correlations (see equation (12)) with its FT being $\widetilde{P}(\xi,\eta) = \widetilde{R}(\eta)\widetilde{Q}(\xi) + \widetilde{C}(\xi,\eta)$, and that 'propagation' of correlations originates from two sources. The first, the RHS of equation (26), is responsible for the 'propagation' of squeezing into the 'initial' correlations $\widetilde{C}(\xi,\eta)$. In the second, equation (27), 'propagation' occurs only for $\phi \neq n\pi$ when additional correlations are created into the 'initial' FT of the uncorrelated part of the HF, $\widetilde{R}(\eta)\widetilde{Q}(\xi)$. In appendix B, the correlations $C^{(1)}$ and $C^{(2)}$ are obtained for the Fock states in the SSR.

4. Fock states in the SSR

The density operator $\boldsymbol{\rho}_n = |n\rangle\langle n|$ mapped in the coherent states representation yields a Poisson distribution [11]

$$P_n(p,q) = |\langle pq|n\rangle|^2 = \frac{1}{n!}\left(\frac{p^2+q^2}{2}\right)^n \exp\left(-\frac{p^2+q^2}{2}\right) \tag{28}$$

with $n = 0, 1, 2, \ldots$. So, the respective FT

$$\widetilde{P}_n(\xi,\eta) = \frac{(-1)^n}{2^{2n}n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\mathcal{H}_{2k}\left(\frac{\xi}{\sqrt{2}}\right)\mathcal{H}_{2(n-k)}\left(\frac{\eta}{\sqrt{2}}\right)\exp\left(-\frac{\xi^2+\eta^2}{2}\right) \tag{29}$$

where $\mathcal{H}_m(x)$ is the Hermite polynomial, represents an initial step for calculating the HF $P_n(p,q;\lambda,\phi)$ in the squeezed states representation. In fact, substituting equation (29) into (5) and evaluating the integrations over ξ and η [12, sections 7.374–8], we get

$$P_n(p,q;\lambda,\phi) = \frac{2\sqrt{\lambda}}{\lambda+1}\left(\frac{\lambda-1}{\lambda+1}\right)^n\frac{1}{2^n n!}\left|\mathcal{H}_n\left(\frac{\lambda q_r + \mathrm{i}p_r}{\sqrt{\lambda^2-1}}\right)\right|^2 \exp\left(-\frac{\lambda q_r^2 + p_r^2}{\lambda+1}\right) \tag{30}$$

where q_r and p_r are the rotated variables defined in section 2. This expression was initially obtained in [13], and later used by Schleich *et al* [1] in the oscillatory behaviour study of the distribution $P_n(p,q;\lambda,0)$. Now, using the mathematical relation [14]

$$|\mathcal{H}_n(z)|^2 = 2^n n!\sum_{k=0}^{n}(-1)^k\mathcal{L}_k^{(-1/2)}(2x^2)\mathcal{L}_{n-k}^{(-1/2)}(-2y^2) \qquad (z = x+\mathrm{i}y)$$

in which $\mathcal{L}_m^{(\alpha)}(x)$ is the associated Laguerre polynomial, equation (30) can be written in an equivalent form:

$$\begin{aligned}P_n(p,q;\lambda,\phi) = {}& \frac{2\sqrt{\lambda}}{\lambda+1}\left(\frac{\lambda-1}{\lambda+1}\right)^n\sum_{k=0}^{n}(-1)^k\mathcal{L}_k^{(-1/2)}\left(\frac{2\lambda^2 q_r^2}{\lambda^2-1}\right)\mathcal{L}_{n-k}^{(-1/2)}\left(-\frac{2p_r^2}{\lambda^2-1}\right)\\ &\times\exp\left(-\frac{\lambda q_r^2+p_r^2}{\lambda+1}\right).\end{aligned} \tag{31}$$

The HF $Q_n(q)$ is obtained with the help of equation (28), i.e.,

$$Q_n(q) = \int_{-\infty}^{\infty}\frac{\mathrm{d}p}{\sqrt{2\pi}}P_n(p,q) = \exp\left(-\frac{q^2}{2}\right)\sum_{k=0}^{n}\mathcal{L}_{n-k}^{(-1/2)}(0)\frac{q^{2k}}{2^k k!} \tag{32}$$

whose FT is given by

$$\widetilde{Q}_n(\xi) = \sum_{k=0}^{n}\mathcal{L}_{n-k}^{(-1/2)}(0)\mathcal{L}_k^{(-1/2)}\left(\frac{\xi^2}{2}\right)\exp\left(-\frac{\xi^2}{2}\right). \tag{33}$$

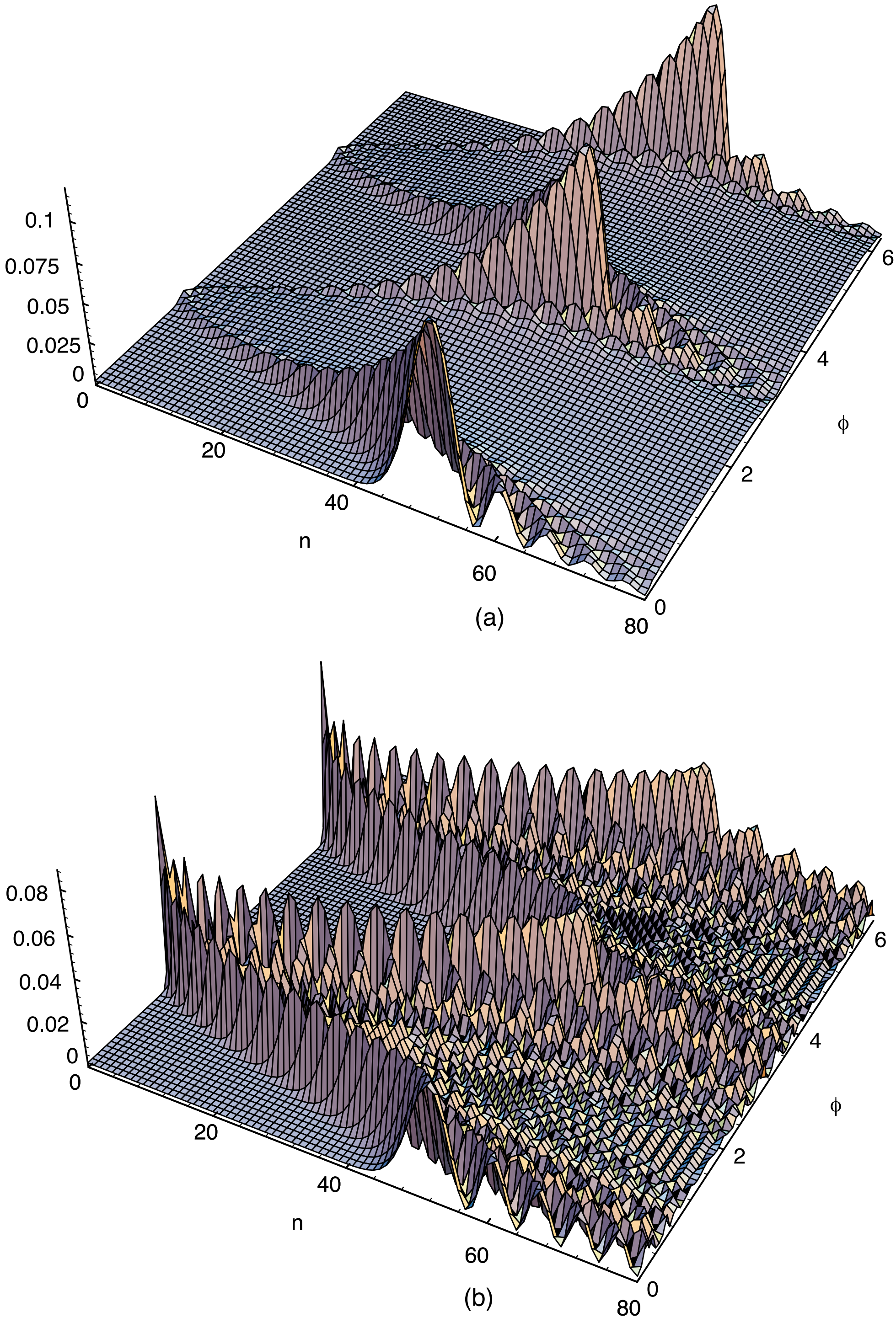

Figure 1. Plots of $P_n(p, q; \lambda, \phi)$ versus n and ϕ with $p^2 + q^2 = 98$. Figures 1(*a*)–(*d*) correspond to $\lambda = 21, 201, \frac{1}{21}$ and $\frac{1}{201}$, respectively.

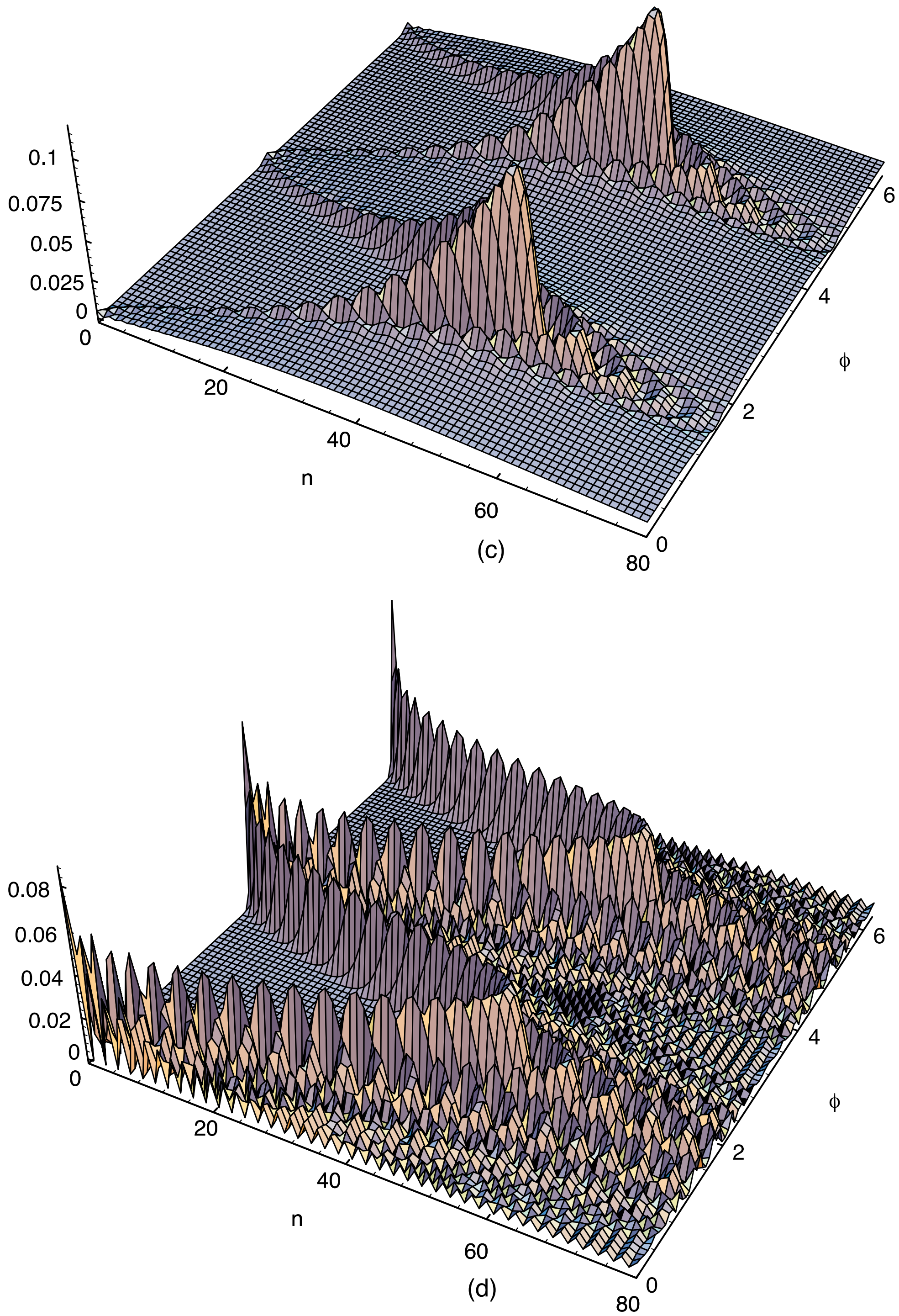

Figure 1. (Continued.)

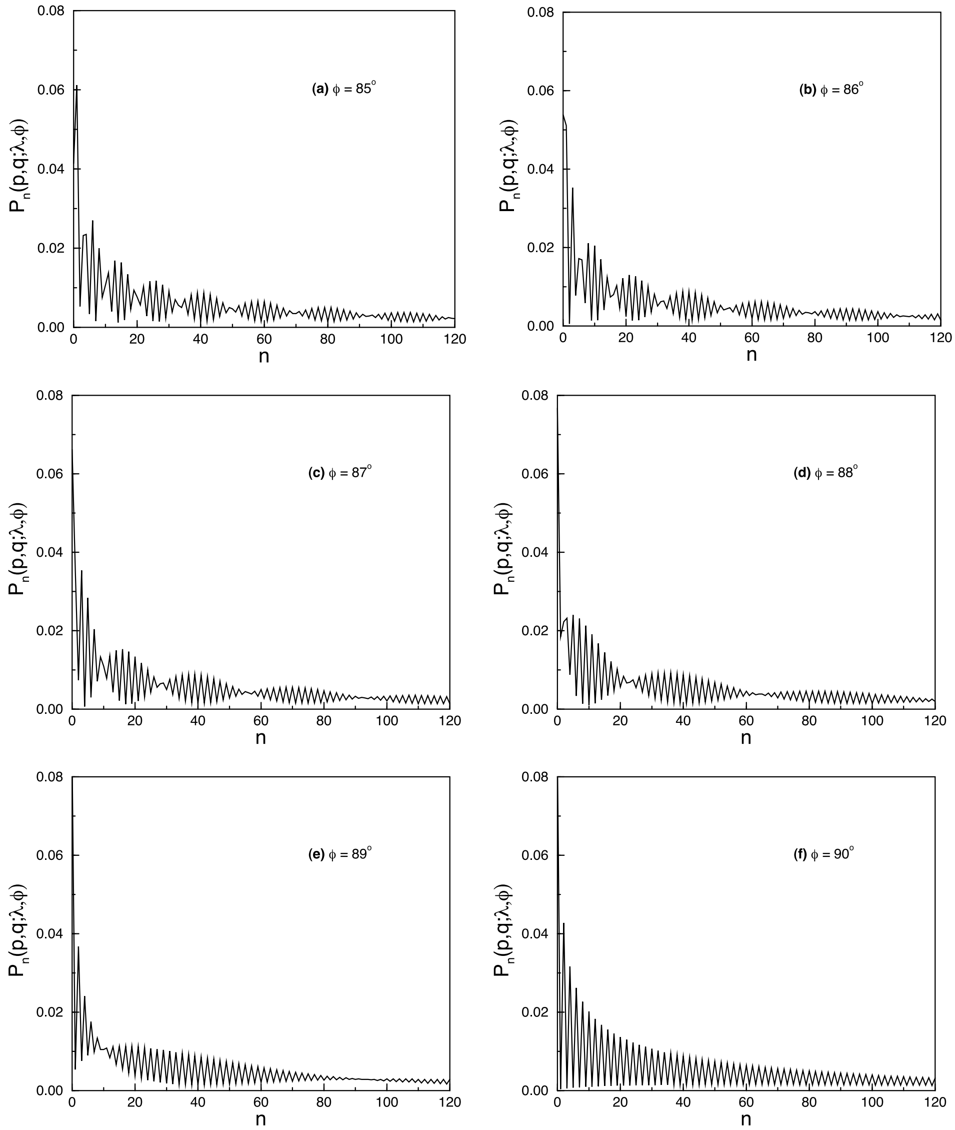

Figure 2. Plots of $P_n(p, q; \lambda, \phi)$ versus n with $\phi = 85°, \ldots, 90°$ and $\lambda = 201$. The phase-space variables are transformed into $q = 7\sqrt{2}\cos\theta$ and $p = 7\sqrt{2}\sin\theta$, which leads to $q_r = 7\sqrt{2}\cos(\theta - \phi/2)$ and $p_r = 7\sqrt{2}\sin(\theta - \phi/2)$. For mapping the parameters used by Dutta *et al* [4] we fixed $\theta = 3\phi/2$.

Substituting this result into equation (15) and performing the integration with respect to ξ, we get

$$Q_n(q; \lambda, \phi) = \sqrt{\frac{2\lambda}{(\lambda+1)[\cos^2(\phi/2) + \lambda\sin^2(\phi/2)]}} \times \sum_{k=0}^{n} (-1)^k \mathcal{L}_{n-k}^{(-1/2)}(0) \left[\frac{\lambda-1}{\lambda+1}\frac{\cos^2(\phi/2) - \lambda\sin^2(\phi/2)}{\cos^2(\phi/2) + \lambda\sin^2(\phi/2)}\right]^k$$

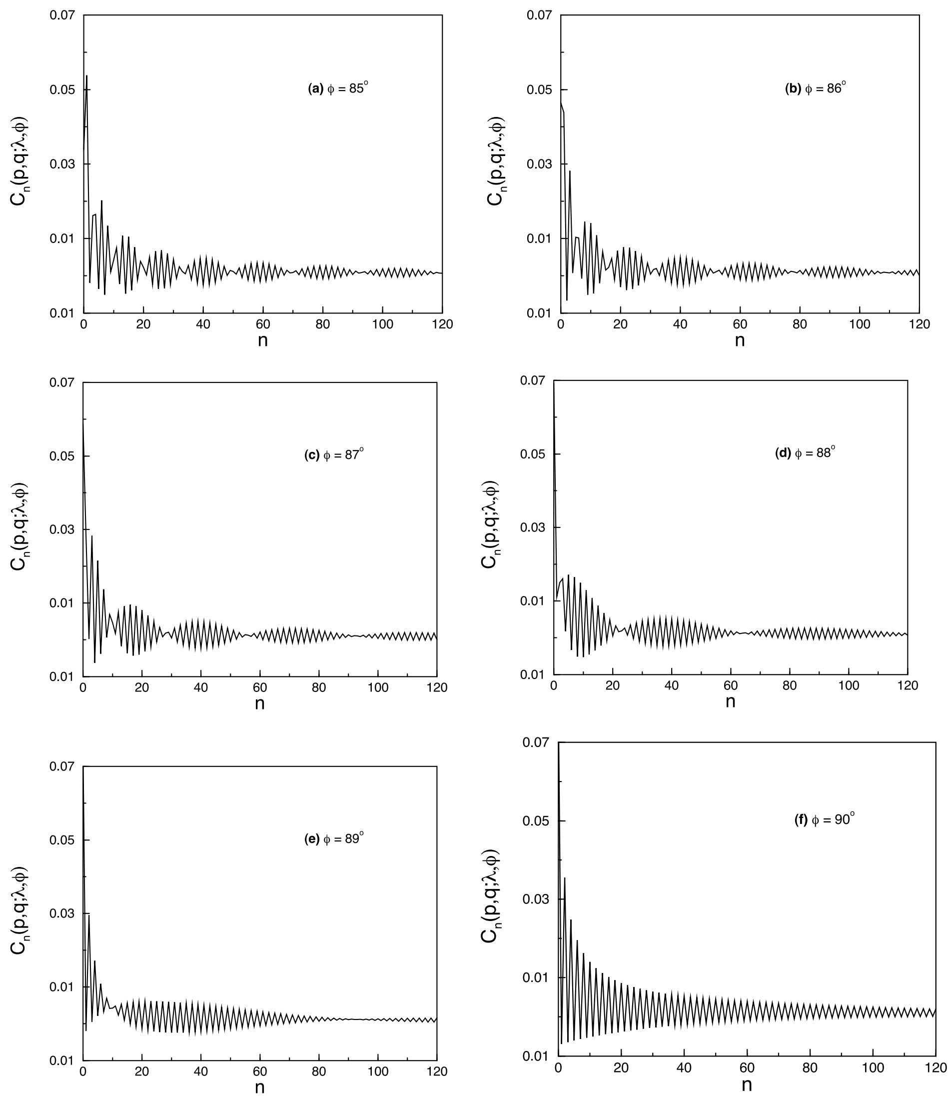

Figure 3. Plots of $C_n(p, q; \lambda, \phi)$ versus n for the same parameters set used in figure 2, where we see the presence of beats again. This fact corroborates the phase-space interference concept since *correlations* and *interference* effects are closely connected.

$$\times\mathcal{L}_k^{(-1/2)}\left[\frac{2\lambda^2 q^2}{(\lambda^2-1)[\cos^4(\phi/2)-\lambda^2\sin^4(\phi/2)]}\right]$$
$$\times\exp\left[-\frac{\lambda q^2}{(\lambda+1)[\cos^2(\phi/2)+\lambda\sin^2(\phi/2)]}\right]. \qquad (34)$$

In order to obtain the MDF $R_n(p; \lambda, \phi)$, we only need the symmetry properties (22),

$$R_n(p;\lambda,\phi) = \sqrt{\frac{2\lambda}{(\lambda+1)[\lambda\cos^2(\phi/2)+\sin^2(\phi/2)]}}$$

$$\times \sum_{k=0}^{n} \mathcal{L}_k^{(-1/2)}(0) \left[\frac{\lambda - 1}{\lambda + 1} \frac{\lambda \cos^2(\phi/2) - \sin^2(\phi/2)}{\lambda \cos^2(\phi/2) + \sin^2(\phi/2)} \right]^{n-k}$$
$$\times \mathcal{L}_{n-k}^{(-1/2)} \left[-\frac{2\lambda^2 p^2}{(\lambda^2 - 1)[\lambda^2 \cos^4(\phi/2) - \sin^4(\phi/2)]} \right]$$
$$\times \exp \left[-\frac{\lambda p^2}{(\lambda + 1)[\lambda \cos^2(\phi/2) + \sin^2(\phi/2)]} \right]. \tag{35}$$

Figures 1(*a*)–(*d*) show the three-dimensional plots of $P_n(p, q; \lambda, \phi)$ versus n and ϕ, for $\lambda = 21, 201, \frac{1}{21}, \frac{1}{201}$, respectively. The plane $\phi = 0$ in figure 1(*a*) corresponds to the oscillations pointed out in [1], which depend strongly on ϕ, showing a periodicity of π. Now, for $\lambda = 201$ (figure 1(*b*)) we observe the occurence of rich structures, although the beats pointed out in [4] cannot be perceived. In fact, they are revealed in figures 2 and 3. Figures 2(*a*)–(*f*) show the plots of P_n versus n for $\phi = 85^\circ, \ldots, 90^\circ$ and $\lambda = 201$, where the beat structure becomes evident; however, it disappears at angles close to 90°. Following the arguments presented in [4] and corroborated by Mandal [7], this beat structure is a consequence of the quantum interference in phase space. Figures 3(*a*)–(*f*) show the plots of C_n versus n for the same parameters used in figures 2(*a*)–(*f*), where the beat structure is present again. This fact connects the *correlations* and *interference* effects in phase space, and provides further insights into the phenomenon. Moreover, we observe similar kinds of plots for $\lambda = \frac{1}{21}$ and $\lambda = \frac{1}{201}$, figures 1(*c*) and (*d*), respectively, where they are now shifted by $\pi/2$.

5. Summary and conclusions

We have considered the HF $P(p, q; \lambda, \phi)$ with emphasis on the marginal and CDFs, showing that all three satisfy the pseudo-diffusion equations if one considers that the squeezing parameter λ plays the role of a time. The solution, obtained from the FT method, permits one to calculate $P(p, q; \lambda, \phi)$, given an 'initial' HF $P(p, q)$ with a kernel $K(\xi, \eta; \lambda, \phi)$ responsible by the propagation of squeezing. The decomposition of the kernel into three factors, equation (23), permits one to write the CDF as a sum of two terms, having different interpretations: the first term, equation (26), is the propagation of 'initial' correlations contained in $P(p, q)$; whereas the second term, equation (27), is responsible for introducing additional correlations into the uncorrelated 'initial' product of the MDFs $Q(q)R(p)$.

Finally, we remind that the formal procedure employed throughout this paper is advantageous if compared with the direct and lengthy calculation exposed in appendix A. In the specific case of the number state, the decomposition of the CDF into two terms should permit one to investigate more thoroughly the origin of beats. Although having attained a formal expression for both (see appendix B), the numerical calculation presents difficulties due to its complexity.

The multimode squeezed-states representation can also be considered within the present formalism. In particular, M Selvadoray *et al* [15] studied the two-mode squeezed-state photon distribution. Again they verified the presence of beats. In this case, the CDF plays a crucial role in the understanding of this effect.

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Appendix A. MDFs for the number states by direct integration

The usual procedure to calculate the MDFs for the number states consists of the integration of equation (30) with respect to the variables p or q, respectively,

$$Q_n(q;\lambda,\phi) = \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{\sqrt{2\pi}} P_n(p,q;\lambda,\phi) \tag{A.1}$$

$$R_n(p;\lambda,\phi) = \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\sqrt{2\pi}} P_n(p,q;\lambda,\phi). \tag{A.2}$$

We calculate (A.1) by direct integration and present some properties inherent to the marginal distributions.

Consider initially, the integral representation of the Hermite polynomial [16]

$$\mathcal{H}_n(z) = (-2\mathrm{i})^n \mathrm{e}^{z^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}u}{\sqrt{\pi}} u^n \mathrm{e}^{-u^2+2\mathrm{i}uz}$$

which permits one to write (30) in a more convenient form:

$$\begin{aligned} P_n(p,q;\lambda,\phi) = \frac{2\sqrt{\lambda}}{\lambda+1}\frac{(-2)^n}{n!}\left(\frac{\lambda-1}{\lambda+1}\right)^n \exp\left\{-\left[\frac{\sin^2(\phi/2)-\lambda\cos^2(\phi/2)}{\lambda-1}\right]q^2\right\} \\ \times \exp\left\{-\left[\frac{\cos^2(\phi/2)-\lambda\sin^2(\phi/2)}{\lambda-1}\right]p^2 + \left(\frac{\lambda+1}{\lambda-1}\sin\phi\right)pq\right\} \\ \times \int_{-\infty}^{\infty} \frac{\mathrm{d}u\,\mathrm{d}v}{\pi}(uv)^n \exp\left\{-(u^2+v^2) + \frac{2}{\sqrt{\lambda^2-1}}[(u-v)\sin(\phi/2)\right. \\ \left. +\mathrm{i}\lambda(u+v)\cos(\phi/2)]q\right\} \\ \times \exp\left\{-\frac{2}{\sqrt{\lambda^2-1}}[(u-v)\cos(\phi/2)-\mathrm{i}\lambda(u+v)\sin(\phi/2)]p\right\}. \end{aligned} \tag{A.3}$$

Substituting (A.3) into (A.1) and integrating with respect to p, we obtain

$$\begin{aligned} Q_n(q;\lambda,\phi) = \sqrt{2\alpha^2}\frac{(-2)^n}{n!}\left(\frac{\alpha}{\beta}\right)^n \mathrm{e}^{-(\alpha q)^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}y}{\sqrt{\pi}} y^n \mathrm{e}^{-(y-\mathrm{i}\alpha q)^2} \\ \times \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{\pi}} x^n \exp\left\{-\left[x - \mathrm{i}\left(\beta q + \mathrm{i}\frac{\beta}{2\alpha}\left(1-\frac{\alpha^2}{\beta^2}\right)y\right)\right]^2\right\} \end{aligned} \tag{A.4}$$

for values of the squeeze parameter in the intervals $1 < \lambda < \cot^2(\phi/2)$ or $\cot^2(\phi/2) < \lambda < 1$, where

$$\alpha = \sqrt{\frac{\lambda}{(\lambda+1)[\cos^2(\phi/2)+\lambda\sin^2(\phi/2)]}}$$

and

$$\beta = \sqrt{\frac{\lambda}{(\lambda-1)[\cos^2(\phi/2)-\lambda\sin^2(\phi/2)]}}.$$

Integration over the variable x leads to [12, sections 3.462–4]

$$\begin{aligned} Q_n(q;\lambda,\phi) = \sqrt{2\alpha^2}\frac{(-\mathrm{i})^n}{n!}\left(\frac{\alpha}{\beta}\right)^n \mathrm{e}^{-(\alpha q)^2} \\ \times \int_{-\infty}^{\infty} \frac{\mathrm{d}y}{\sqrt{\pi}} y^n \mathcal{H}_n\left[\beta q + \mathrm{i}\frac{\beta}{2\alpha}\left(1-\frac{\alpha^2}{\beta^2}\right)y\right] \mathrm{e}^{-(y-\mathrm{i}\alpha q)^2}. \end{aligned} \tag{A.5}$$

Now, using the relation [14]

$$\mathcal{H}_n(z+w)=\sum_{k=0}^{n}\mathcal{L}_{n-k}^{(k)}(0)(2w)^{n-k}\mathcal{H}_k(z)$$

for the Hermite polynomial present in (A.5),

$$\mathcal{H}_n\left[\beta q+\mathrm{i}\frac{\beta}{2\alpha}\left(1-\frac{\alpha^2}{\beta^2}\right)y\right]=\sum_{k=0}^{n}\mathcal{L}_{n-k}^{(k)}(0)\left[\mathrm{i}\frac{\beta}{\alpha}\left(1-\frac{\alpha^2}{\beta^2}\right)y\right]^{n-k}\mathcal{H}_k(\beta q)$$

we get

$$Q_n(q;\lambda,\phi)=\frac{\sqrt{2\alpha^2}}{2^n n!}\left(\frac{\alpha}{\beta}\right)^n \mathrm{e}^{-(\alpha q)^2}\sum_{k=0}^{n}\mathcal{L}_{n-k}^{(k)}(0)\left[\frac{\beta}{2\alpha}\left(\frac{\alpha^2}{\beta^2}-1\right)\right]^{n-k}\mathcal{H}_k(\beta q)\mathcal{H}_{2n-k}(\alpha q). \tag{A.6}$$

Note that (A.6) can be written in a compact form equivalent to equation (34), i.e., in terms of the associated Laguerre polynomial. For this purpose it is necessary to verify the equality

$$\frac{1}{2^{2n}n!}\sum_{s=0}^{n}\mathcal{L}_{n-s}^{(s)}(0)\left(\frac{2\alpha}{\beta}\right)^s\left(\frac{\alpha^2}{\beta^2}-1\right)^{n-s}\mathcal{H}_s(\beta q)\mathcal{H}_{2n-s}(\alpha q) = \sum_{k=0}^{n}c_{nk}(\alpha,\beta)\mathcal{L}_k^{(-1/2)}[2(\alpha\beta q)^2] \tag{A.7}$$

and to determine the coefficients $c_{nk}(\alpha,\beta)$.

The relation established by Bailey [17] for the product of Hermite polynomials,

$$\mathcal{H}_m(ax)\mathcal{H}_l(bx)=\sum_{j=0}^{[\frac{m+l}{2}]}(-1)^j\frac{m!}{j!(m-2j)!}\frac{a^{m-2j}b^{2j-l}}{(\sqrt{a^2+b^2})^{m-l}} \times {}_2\mathcal{F}_1\left(m+1,-l;m-2j+1;\frac{a^2}{a^2+b^2}\right)\mathcal{H}_{m+l-2j}(\sqrt{a^2+b^2}x)$$

where ${}_2\mathcal{F}_1(a_1,a_2;a_3;z)$ is the hypergeometric function, permits one to verify equality (A.7) through

$$\mathcal{H}_s(\beta q)\mathcal{H}_{2n-s}(\alpha q)=\frac{(-1)^n(2n-s)!}{2^{n-s}(\alpha\beta)^{2n-s}}\sum_{k=0}^{n}\frac{k!}{(2k-s)!(n-k)!}(2\alpha)^{2k}\beta^{2(n-k)} \times{}_2\mathcal{F}_1\left(2n-s+1,-s;2k-s+1;\frac{1}{2\beta^2}\right)\mathcal{L}_k^{(-1/2)}[2(\alpha\beta q)^2] \tag{A.8}$$

and to determine the coefficients $c_{nk}(\alpha,\beta)$,

$$c_{nk}(\alpha,\beta)=\frac{(-1)^n}{2^{2n}}\frac{k!}{(n-k)!}\frac{(2\alpha)^{2k}\beta^{2(n-k)}}{(2\alpha^2\beta^2)^n}\sum_{s=0}^{n}\frac{(2n-s)!}{s!(n-s)!(2k-s)!}(2\alpha)^{2s}\left(\frac{\alpha^2}{\beta^2}-1\right)^{n-s} \times{}_2\mathcal{F}_1\left(2n-s+1,-s;2k-s+1;\frac{1}{2\beta^2}\right). \tag{A.9}$$

In fact, the sum present in (A.9) can be performed since we used the following relation for the hypergeometric function [14]:

$${}_2\mathcal{F}_1\left(2n-s+1,-s;2k-s+1;\frac{1}{2\beta^2}\right) = \frac{s!(2k-s)!}{(2n-s)!}\sum_{l=0}^{s}(-1)^l\frac{(2n-s+l)!}{(s-l)!(2k-s+l)!}\frac{(2\beta^2)^{-l}}{l!}. \tag{A.10}$$

Then, substituting (A.10) into (A.9) and calculating the sums, we obtain a simple expression for the coefficients

$$c_{nk}(\alpha,\beta) = (-1)^k \frac{[2(n-k)-1]!!}{[2(n-k)]!!}\left(\frac{\alpha}{\beta}\right)^{2k} = (-1)^k \mathcal{L}_{n-k}^{(-1/2)}(0)\left(\frac{\alpha}{\beta}\right)^{2k}. \tag{A.11}$$

So, the marginal distribution (A.6) can be expressed in a form equivalent to equation (34). Adopting an analogous procedure for equation (A.2) we obtain (35); however, the values of the squeezing parameters are restricted in the intervals $\tan^2(\phi/2) > \lambda > 1$ or $\tan^2(\phi/2) < \lambda < 1$.

In addition to the symmetry relations established in section 3 (see equation (22)), the MDFs exhibit the following properties:

$$\text{(i)}\quad \lim_{\lambda\to\infty} Q_n(q;\lambda,0) = |\Psi_n(q)|^2 \qquad \text{and} \qquad \lim_{\lambda\to 0} R_n(p;\lambda,0) = |\Phi_n(p)|^2 \tag{A.12}$$

$$\text{(ii)}\quad \sum_{n=0}^{\infty} Q_n(q;\lambda,\phi) = \sum_{n=0}^{\infty} R_n(p;\lambda,\phi) = \sum_{n=0}^{\infty} \mathcal{L}_n^{(-1/2)}(0) \to \infty. \tag{A.13}$$

The first property does not characterize the HF as a probability distribution but only emphasizes the character of quasiprobability distribution: limit values of λ recover the squared moduli of wavefunctions [10, 18]. With respect to the second property, it is a direct consequence of the scalar product for the squeezed states, i.e. $\sum_{n=0}^{\infty} P_n(p,q;\lambda,\phi) = 1$. In fact, the divergence is a consequence of the integration step of this relation over the variables p or q. Now, considering the normalization of equation (30), $\int_{-\infty}^{\infty} \frac{\mathrm{d}p\,\mathrm{d}q}{2\pi} P_n(p,q;\lambda,\phi) = 1$, we obtain the third property:

$$\text{(iii)}\quad \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\sqrt{2\pi}} Q_n(q;\lambda,\phi) = 1 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{\sqrt{2\pi}} R_n(p;\lambda,\phi) = 1. \tag{A.14}$$

Appendix B. The components of the CDF

The components of the CDF $C(p,q;\lambda,\phi)$ can also be written as

$$C^{(1)}(p,q;\lambda,\phi) = P(p,q;\lambda,\phi) - C^{(3)}(p,q;\lambda,\phi) \tag{B.1}$$

$$C^{(2)}(p,q;\lambda,\phi) = C^{(3)}(p,q;\lambda,\phi) - Q(q;\lambda,\phi)R(p;\lambda,\phi) \tag{B.2}$$

with

$$C^{(3)}(p,q;\lambda,\phi) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\xi\,\mathrm{d}\eta}{2\pi} \mathrm{e}^{\mathrm{i}(\eta p - \xi q)} K(\xi,\eta;\lambda,\phi)\widetilde{Q}(\xi;1)\widetilde{R}(\eta;1). \tag{B.3}$$

Present in both equations (B.1) and (B.2), the term $C^{(3)}$ is responsible for introducing correlations into the marginal distributions product through the relation

$$C^{(3)}(p,q;\lambda,\phi) = \exp\left[\left(\frac{\lambda^2-1}{4\lambda}\sin\phi\right)\frac{\partial^2}{\partial p\partial q}\right] Q(q;\lambda,\phi)R(p;\lambda,\phi) \tag{B.4}$$

where we used the result

$$\mathrm{e}^{\mathrm{i}(\eta p-\xi q)} K(\xi,\eta;\lambda,\phi) = \exp\left[\left(\frac{\lambda^2-1}{4\lambda}\sin\phi\right)\frac{\partial^2}{\partial p\partial q}\right]\mathrm{e}^{\mathrm{i}(\eta p-\xi q)} k_Q(\xi;\lambda,\phi)k_R(\eta;\lambda,\phi).$$

For $\phi = 0$, equations (B.1)–(B.3) simplify to

$$C^{(1)}(p,q;\lambda,0) = C(p,q;\lambda,0)$$
$$C^{(2)}(p,q;\lambda,0) = 0$$
$$C^{(3)}(p,q;\lambda,0) = Q(q;\lambda,0)R(p;\lambda,0).$$

In this appendix we calculate the component $C^{(3)}$ for the number states and, consequently, the components $C^{(1)}$ and $C^{(2)}$ can be totally determined.

In order to simplify the calculations, let us initially consider the MDFs and CDFs expressed in terms of the parameters α_1, α_2 and α_3:

$$Q_n(q;\lambda,\phi) = \sqrt{2\alpha_1^2}\,\mathrm{e}^{-(\alpha_1 q)^2}\sum_{k=0}^{n}\mathcal{L}_{n-k}^{(k)}(0)(-2\alpha_1^2)^k\mathcal{L}_k^{(-1/2)}[(\alpha_1 q)^2] \tag{B.5}$$

$$R_n(p;\lambda,\phi) = \sqrt{2\alpha_2^2}\,\mathrm{e}^{-(\alpha_2 p)^2}\sum_{k=0}^{n}\mathcal{L}_{n-k}^{(k)}(0)(-2\alpha_2^2)^k\mathcal{L}_k^{(-1/2)}[(\alpha_2 p)^2] \tag{B.6}$$

$$C_n^{(3)}(p,q;\lambda,\phi) = \sum_{l=0}^{\infty}\frac{\alpha_3^l}{l!}\frac{\partial^{2l}}{\partial p^l\partial q^l}Q_n(q;\lambda,\phi)R_n(p;\lambda,\phi) \tag{B.7}$$

with

$$\alpha_1 = \sqrt{\frac{\lambda}{(\lambda+1)[\cos^2(\phi/2)+\lambda\sin^2(\phi/2)]}}$$

$$\alpha_2 = \sqrt{\frac{\lambda}{(\lambda+1)[\lambda\cos^2(\phi/2)+\sin^2(\phi/2)]}}$$

$$\alpha_3 = \frac{\lambda^2-1}{4\lambda}\sin\phi$$

and $\alpha_1^2+\alpha_2^2 = 2\alpha_1^2\alpha_2^2$. Expressions (B.5) and (B.6) are the alternative way of writing the marginal distributions (34) and (35), respectively, since we used the properties of the associated Laguerre polynomials. Moreover, using the relation

$$\frac{\partial^l}{\partial x^l}\{\mathrm{e}^{-(ax)^2}\mathcal{L}_k^{(-1/2)}[(ax)^2]\} = \frac{(-1)^k}{2^{2k}k!}(-a)^l\mathrm{e}^{-(ax)^2}\mathcal{H}_{l+2k}(ax)$$

these expressions permit one to calculate the component $C_n^{(3)}$, given the result

$$C_n^{(3)}(p,q;\lambda,\phi) = 2\alpha_1\alpha_2\mathrm{e}^{-[(\alpha_1 q)^2+(\alpha_2 p)^2]}\sum_{k=0}^{n}\mathcal{L}_{n-k}^{(k)}(0)\frac{\alpha_1^{2k}}{2^k k!}\sum_{m=0}^{n}\mathcal{L}_{n-m}^{(m)}(0)\frac{\alpha_2^{2m}}{2^m m!}$$
$$\times\sum_{l=0}^{\infty}\frac{(\alpha_1\alpha_2\alpha_3)^l}{l!}\mathcal{H}_{l+2k}(\alpha_1 q)\mathcal{H}_{l+2m}(\alpha_2 p). \tag{B.8}$$

The infinity sum present in (B.8) can be performed by using the formula [19, section 5.12.2.1]

$$\sum_{k=0}^{\infty}\frac{t^k}{k!}\mathcal{H}_{k+m}(x)\mathcal{H}_{k+n}(y) = \frac{1}{\sqrt{(1-4t^2)^{m+n+1}}}\exp\left[\frac{4txy-4t^2(x^2+y^2)}{1-4t^2}\right]$$
$$\times\sum_{r=0}^{\{m,n\}}r!\mathcal{L}_{m-r}^{(r)}(0)\mathcal{L}_{n-r}^{(r)}(0)(4t)^r\mathcal{H}_{m-r}\left(\frac{x-2ty}{\sqrt{1-4t^2}}\right)\mathcal{H}_{n-r}\left(\frac{y-2tx}{\sqrt{1-4t^2}}\right)|t|<\tfrac{1}{2}$$

where $\{m,n\}$ stands for the minor of m and n, which leads to

$$C_n^{(3)}(p,q;\lambda,\phi) = \frac{2\alpha_1\alpha_2}{\sqrt{1-(2\alpha_1\alpha_2\alpha_3)^2}}\exp\left[-\frac{\alpha_1^2q^2-4\alpha_1^2\alpha_2^2\alpha_3 pq+\alpha_2^2p^2}{1-(2\alpha_1\alpha_2\alpha_3)^2}\right]$$
$$\times\sum_{k=0}^{n}\frac{\mathcal{L}_{n-k}^{(k)}(0)}{2^k k!}\left[\frac{\alpha_1^2}{1-(2\alpha_1\alpha_2\alpha_3)^2}\right]^k\sum_{m=0}^{n}\frac{\mathcal{L}_{n-m}^{(m)}(0)}{2^m m!}\left[\frac{\alpha_2^2}{1-(2\alpha_1\alpha_2\alpha_3)^2}\right]^m$$
$$\times\sum_{r=0}^{\{2k,2m\}}r!\mathcal{L}_{2k-r}^{(r)}(0)\mathcal{L}_{2m-r}^{(r)}(0)(4\alpha_1\alpha_2\alpha_3)^r$$
$$\times\mathcal{H}_{2k-r}\left(\frac{\alpha_1 q-2\alpha_1\alpha_2^2\alpha_3 p}{\sqrt{1-(2\alpha_1\alpha_2\alpha_3)^2}}\right)\mathcal{H}_{2m-r}\left(\frac{\alpha_2 p-2\alpha_1^2\alpha_2\alpha_3 q}{\sqrt{1-(2\alpha_1\alpha_2\alpha_3)^2}}\right). \tag{B.9}$$

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